# The Mathematics Teacher

DEVOTED TO THE INTERESTS OF MATHEMATICS IN JUNIOR AND SENIOR HIGH SCHOOLS

VOLUME XVI    NOVEMBER, 1923    No. 7

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# THE MATHEMATICS TEACHER

VOLUME XVI — NOVEMBER, 1923 — NUMBER 7

## MATHEMATICS CLUB PROGRAM

By ALBERT HARRY WHEELER
North High School, Worcester, Mass.

A committee of the Association of Teachers of Mathematics of New England has prepared a number of programs for mathematics clubs which will appear in THE MATHEMATICS TEACHER in a series of articles during the year 1923-1924. The service has been undertaken for the purpose of providing lists of articles and books on various topics of interest to teachers who in many cases have little time to make the search themselves. The magazines published by the mathematical societies during the past twenty years have been examined and such articles listed as have seemed to have especial interest or value.

Professor Lennie P. Copeland of Wellesley College has directed the work and as chairman of the committee has personally attended to the details of arranging the results. The material has been listed by Subject, Author, Publisher with date of publication and number of pages, and has been graded by the committee as in their judgment it was fair, good or excellent.

The references given are not exhaustive, and doubtless many excellent articles have been overlooked, but it is hoped that the work may prove to be of value to teachers and students alike. It is believed that many teachers would undertake to conduct mathematics clubs if they could secure references classified as to subject matter relating to selected topics. In addition to making club programs, several lists of books and magazines helpful to teachers have been collected.

Besides THE MATHEMATICS TEACHER the following publications have been searched for articles: *The American Mathematical Monthly, School Science and Mathematics, Science, Nature, The Youth's Companion, Scientific American Supplement*, and other magazines as the *Forum*, etc. Many articles are to be found in the encyclopedias and in such works as those of Ball, Dudeney, Schubert, Ahrens, Leitzmann, and others who have written on mathematical recreations and curiosities. Each pro-

gram has many more references than will be used at any one time by a club, but among those given there is opportunity for selection.

Among the topics which will appear during the year are the following:

1. Number and Measuring Systems.
   Duodecimal, Sexagesimal, Binary; Metric, Digital, Indian, Chinese, Maya.
2. Ancient Arithmetic.
   Numerals, Counting, Origin of Names and Symbols.
3. Some Interesting Numbers.
   $\pi$, $i$, Perfect, Amicable, Huge Numbers, Mysterious Numbers.
4. Ancient Geometry.
   Egyptian, Chinese, Greek, Dynamic Symmetry.
5. Geometrical Constructions and Measurements.
   Paper Folding, Double Edged Ruler, 17-gon.
6. Origin and Development of Algebra.
   Egyptian, Babylonian, Indian Symbols, etc.
7. Famous Problems.
   Magic Squares and Cubes, Russian Multiplication, Trisecting Angle, Squaring Circle, Cattle Problem, Recreations, etc.
8. Some Mathematicians, including Prodigies.
9. Mathematical Instruments.
   Sundials, Calculating Machines, Slide Rules, etc.
10. Applied Mathematics.
    To Science, Business, Cultural Value, Codes, Maps, Calendars.
11. Advanced Mathematics.
    Trigonometry, Limits, Permutations, Charts, Logarithms, Determinants.
12. The Fourth Dimension.
13. Relativity.
14. Non-Euclidean Geometry.
15. Five Mathematical Plays.

## Program 1

### Number and Measuring Systems

I. How various people have counted.
   Beginnings of Counting, L. L. Conant. Sci. Am. S. 60: 24822-4, S. 9, '05. (e.)

Art of Counting, M. de Villiers. Westm. 169: 324-32, Mr. '08. (e.)

Digital Reckoning of the Ancients, L. J. Richardson. Am. Math. M. 23:7-13, Jan. '15. (e.)

Number Systems of the North American Indians, W. C. Eells. Am. Math. M. 20:263-272, 293-299. Nov., Dec., '13. (e.)

Ancient Duodecimal System. Babylonian Conception of Number and Measure R. P. Williams. Sci. Am. S. 68:83. Ag., '07, '09. (e.)

An Ancient Duodecimal System, R P. Williams. Sch. Sci. and Math., 9:516-521. Je., '09. (g.)

Duodecimal System, R. C. Eldridge. Sci. Am., 88:335. My., 2, '03. (g.)

Sexagesimal System and the Division of the Circle, G. A. Miller. Science, n. s. 31:431-2. Mr. 18, '10. (f.)

Decimal System of Numbers, L. C. Karpinski. Pop. Sci. 75: 490-8, N., '09. (e.)

The Binary Scale of Notation, Topics for Club Programs. Am. Math. M. 25:139-142. Mr., '18. (g.)

Not Ten but Twelve, W. B. Smith. Science, n. s. 50:239. S. 12, '19. (e.)

II. The Metric System.

A Brief Historical Consideration of the Metric System, C. Karpinski. Science, n. s. 53:156-7. F. 18, '21. (f.)

Practical Use of the Metric System, L. W. Turner. Sch. Sci. and Math. 7:8-10. Jan., '07. (f.)

Adopt the Metric System, H. N. Kauffman. Sch. Sci. and Math. 19:82-4. Jan., '19. (g.)

The Metric System, Sci. Am. 121:9 Jul. 5, 57 Jul. 19, 157 Ag. 16, 205 Ag. 30, 281 S. 20, '19. (f.)

The Metric System, Sci. Am. 122:111 Jan 31, '20; 123:119 Ag. 7, '20. (f.)

The Metric System, H. T. Wade. Sci. Am. 123:125 Ag., '07, '20. (g.)

The Metric System, A. E. Kennelly, W. C. Wells, H. V. Arny, F. R. Drake, A. W. Miller, G. F. Kung. Sci. M. 4:193-219 Mr., '17. (e.)

Program 2

Arithmetic

I. History.

Evolution of Figures from Ancient Tally Marks. Cur. Lit. 48:289-90. Mr., '10. (g.)

Hindu Arabic Numerals, L. C. Karpinski. Science n. s. 35: 969-70. Je., 21, '12. (g.)

Hindu Arabic Numerals, Smith and Karpinski. Book Review, Science n. s. 35:501-4. Mr. 29, '12. (e.)

Hindu Arabic Numerals, E. R. Turner. Pop. Sci. 81:601-13. D., '12. (g.)

Common Numerals, G. A. Miller. Science n. s. 49:215. F. 28, '19. (f.)

Origin of Our Numerals, F. Cajori. Sci. M. 9:458-64. N., '19. (e.)

Plus and Minus, G. B. Halsted. Science n. s. 37:836-7. My. 30, '13. (g.)

Plus and Minus Again, F. Cajori. Science n. s. 38:51-2. Jul. 11, '13. (g.)

Origin of the Names of Arithmetical and Geometrical Proportion, F. Cajori. Sch. Sci. and Math. 22:734-7. N., '22. (g.)

The Decimalization of Arithmetic. Sch. Sci. and Math. 8: 409-10. My., '08. (f.)

La Disme of Simon Stevin, Vera Sanford. Math. T. 14: 321-33. O., '21. (e.)

Recent Symbolism for Decimal Fractions, F. Cajori. Math. T. 16:183-7. Mr., '23. (e)

II. Old Arithmetics.

The Oldest Mathematical Work Extant, Topics for Club Programs. Am. Math. M. 25:36-7. Jan., '18. (f.)

The Mathematical Handbook of Ahmes, G. A. Miller. Sch. Sci. and Math. 5:567-75. O., '05. (e.)

Mathematics and Idolatry, G. A. Miller. Sch. Sci. and Math. 11:60-3. Jan., '11. (e.)

The Foremost Intellectual Achievement of Ancient America, S G. Morley. Nat. Geog. Mag. 41:109-30. F., '22. (e.)

The Zero and Principle of Local Value Used by the Maya of Central America, F. Cajori. Science n. s. 44:714-7. N. 17, '16. (e.)

The First Work on Mathematics Printed in the New World, D. E. Smith. Am. Math. M. 28:10-5. Jan., '21. (g.)

A History of Arithmetic in the United States, F. Molesworth. Sch. Sci. and Math. 17:854-6. D., '17. (g.)

The Mathematical Sciences in the Latin Colonies of America, F. Cajori. Sci. M. 16:194-204. Feb., '23. (e.)

## Program 3

### Concerning Numbers

I. Some Remarkable Numbers.

A History of $i = \sqrt{-1}$, Lena Vaughn. Sch. Math. 1:173-5. Jan., '04. (g.)

The Evolution of $\pi$, A. J. Schwartz. Sch. Sci. and Math. 11: 791-4. N., '11. (g.)

The Number $\pi$. Topics for Club Programs. Am. Math. M. 26:209-12. My., '19. (f.)

Mathematics in Verse. ($\pi$ to 13 decimals.) Ind. 77:312. Mr., 2, '14. (f.)

Perfect and Amicable Numbers, L. E. Dickson. Sci. M. 12: 349-54. Ap., '21. (e.)

On the Representation of Large Numbers, A. Emch. Sci. M. 2: 272-8. Mr., '16. (g.)

Huge Numbers, Ed. by B. F. Finkel. Am. Math. M. 28:393-4. O., '21. (g.)

II. Curious Properties and Beliefs.

The Favorite Numbers of the Zuni, E. C. Parsons. Sci. M. 3: 596-600. D., '16. (g.)

Mystery of Numbers, W. C. Cornwell. Forum 68:784-90. S., '22. (g.)

The Romantic Aspect of Numbers, S. E. Slocum. Sci. M. 7: 68-79. Jul., '18. (e.)

Queer Beliefs About the Number Seven. Sci. Am. S. 22997 Jul., 4, '03. (e.)

Properties of Numbers, L. C. Karpinski. Sci. Am. 100:243. Mr. 27, '09. (e.)

Properties of Numbers, D. M. Morris. Sci. Am. 100:279. Ap. 10, '09. (e.)

Curiosity of Numbers, A. R. Gallatin, F. D. Mitchell, F. Newcomb. Sci. Am. 98:311, My., 2; 99:9 Jul., 4, '08; 100: 202, Mr., 13, '09. (e.)

Curious Facts About Numbers, A. A. Laughlin, B. Dawson, H. M. Russell, F. Newcomb, J. F. Springer. Sci. Am. 98: 222-3, Mr., 28; 311 My., 2; 371 My., 23; 391 My., 30; 99:299 O., 31; 354-5 N., 21, '08. (e.)

Curious Properties of Numbers, A. Y. Smith, W. B. Pierce. Sci. Am. S. 67:271 Ap., 24; 379 Je., 12, '09. (e.)

Arithmetical Curiority. Sci. Am. S. 77:391. Je., 20, '14. (e.)

Curious Number Puzzles, J. F. Springer. Sci. Am. 101:391. N. 27, '09. (e.)

The Magic Number Cards, R M. Mathews. Sch. Sci. and Math. 13:819-20. D., '13. (g.)

Program 4

Geometry

The Parallel Development of Mathematical Ideas Numerically and Geometrically, L. C. Karpinsky. Sch. Sci. and Math. 20:821-8. D., '20. (e.)

Final Report of National Committee of Fifteen on Geometry Syllabus, Section A. Historical Introduction. Math. T. 5: 48-75. D., '12. (f.)

Foundation of Geometry, A. d'Abro. Sci. Am. 124-67. Jan., 22, '21. (g.)

Our Geometry in Egypt and China, W. A. Austin. Math. T. 16: 78-86. F., '23. (g.)

The Engineers of Ancient Egypt, G. A. McWilliams. Sci. Am 124:132. F. 12, '21. (g.)

The King's Chamber, F. J. Dick. Am. Math. M. 27:262-3. Je., '20. ((f)

Dynamic Symmetry. Sci. Am. M. 4:23-8. Jul.-Oct., '21. (e.)

The Most Pleasing Rectangle, A. A. Bennett. Am. Math. M. 30: 27-30. Jan., '23. (g.)

Certain Undefined Elements and Tacit Assumptions in the First Book of Euclid's Elements, H. E. Webb. Math. T. 12:41-60. D., '19. (f.)

Geometry of the Hindus Sci. Am. S. 62:258-9. D. 22, '06. (e.)

# CRAIG'S EDITION OF EUCLID: ITS "USE AND APPLICATION" OF THE PRINCIPAL PROPOSITIONS GIVEN

By AGNES G. ROWLANDS
Instructor in Methods of Teaching
Jamaica Training School, New York City

In 1818 in Baltimore there was published an edition of Euclid edited by John D. Craig, which contains many practical applications of the theorems in Euclid, which teachers of geometry would find suggestive. Many of them are very familiar ones. To give teachers an idea of the book, I quote a passage from the introduction ~~and a few~~ of the problems.

"Long experience has convinced the editor that the refined abstract reasoning of Simson and Playfair is in a great measure unintelligible to many youths, who are, at the same time capable of understanding the elements of geometry, when presented in a more familiar language.

Another great advantage of Craig's edition it is presumed will be found to result from the "use and application" of the principal propositions given in this work; as they must afford strong inducements to the students to persevere in an arduous

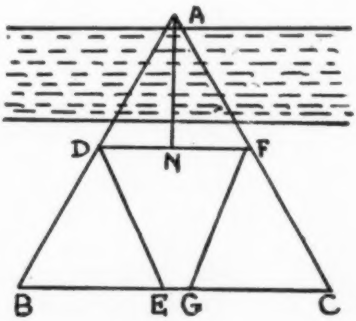



FIGURE 1

undertaking, which otherwise might appear inapplicable to any useful purpose in the affairs of life.

Some of the theorems of Euclid, with "use and application" of them as given by Craig follow:

To describe an equilateral triangle on any given line.

*Use*: Serviceable for measuring an inaccessible line; as, for example lines *AB*, which by reason of a river cannot be measured.

Make a small equilateral triangle *BDE* of wood or copper; and having placed it horizontally on *B*, observe point *A* by side *BD*, and any other point *C* by side *BE*. Then transfer your triangle along line *BC*, and place it upon divers parts of the same line; until at length you find a point *C*, upon which placing the *E* you shall see the point *B*, by the side *CG*, and then point *A* by the side *CF*. The lines *CB* and *CA* (or *AC*) are equal; so by measuring the lines *BC*, you may know the lines *AB*.

2. *Theorem*. When two parallel lines are cut by a transversal, the alternate interior angles are equal; the corresponding angles are equal; the interior angles on the same side of the transversal are supplementary.

*The Use*: Eratosthenes found out by these propositions a way of measuring the circumference of the earth. In order to do this he supposed two rays proceeding from the center of the sun to two points of the earth, to be physically parallel; and also that at Syene, a town in the higher parts of Egypt, the sun came exactly to the zenith on the day of the solstice, observing the wells there to be illuminated to the very bottom. He likewise computed the distance between Alexandria and Syene in stadia.

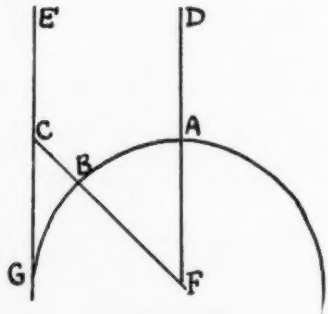



FIGURE 2

Let us, therefore, suppose Syene to be at the point *A* and Alexandria at *B* where we erect a style *BC* perpendicular to the horizon, and let the two lines *DF* and *EG* represent two rays proceeding from the center of the sun upon the day of the solstice which are parallel to each other, *DA*, which passes by

Syene, is perpendicular, that is, it passes through the center of the earth. Having observed by the perpendicular style *BC* the angle *GCB,* made by the ray of the sun *EG*, the rays *DA* and *EG* being parallel, the alternate angles *GCB* and *BFA* are equal: by which means we have the angle *AFB,* and its measure *AB,* which gives us in degrees the distance between Alexandria and Syene. Having supposed this distance to be known in miles the circumference of the earth could be found.

3. Two lines drawn toward the same parts from the extremities of two other lines that are equal and parellel, are also themselves equal and parallel.

*Use*: For measuring perpendicular heights *AG* of mountains, and also their horizontal lines *CG*, which are hid by their bulk.

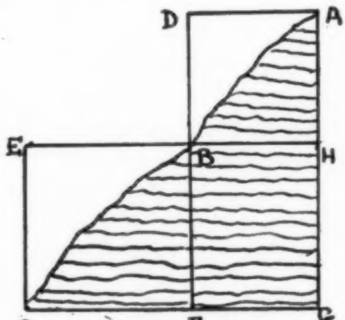



FIGURE 3

Take a large square *ADBH* and place it so at point *A* that side *DB* may fall perpendicularly. Measure sides *AD* and *DB*. Do same again at point B and measure *BE* and *EC*; the sides parallel to horizon *AD* and *BE*, added together give horizontal line *CG*; and the perpendicular sides *DB* and *EC* give height *AG*. This way of measuring is called cultillation.

4. An exterior angle of a triangle is equal to the sum of the two remote interior angles.

*Use*: Used by astronomers in determining parallax (that is, the angle at a heavenly body, when viewed from two different positions).

Suppose point *A* to be the center of the earth, and from point *B,* upon the surface be taken angle *DBC,* that is to say the distance of the star from the zenith *D*. Angle *CBD* minus angle

*BAC* equals angle *C* (the parallax). Astronomical tables show how far remote from the zenith the star should appear to him that should be at the center of the earth (that is they give angle *DAC*).

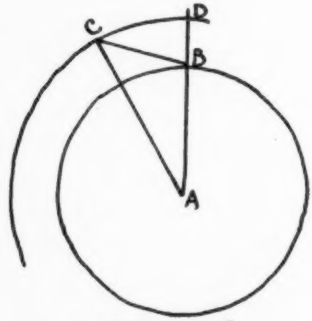


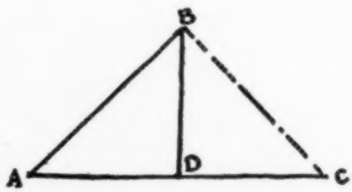

FIGURE 4

The pythagorean theorem as a foundation of trignometry.

Applications in trigonometry.

In a right triangle a perpendicular from the vertex of the right angle to the hypotenuse of the right angle is a mean proportional between the segments of hypotenuse.

FIGURE 5




*Use*: To measure an inaccessible distance by means of a carpenter's square.

If I wish to measure distance *DC*. Measure perpendicular *BD*. Place square upon point *B* so that by one of its sides *BC*, point *C* may be observed, and by the other point *A*. $AD:BD$ equals $BD:DC$. Multiplying *DB* by itself and dividing by *AD*, the quotient would be *DC*.

2. Two triangles are equal if two sides and the included angle of one are respectively equal to two sides and the included angle of the other.

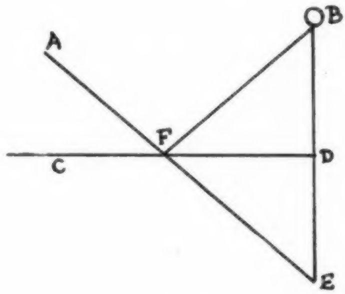
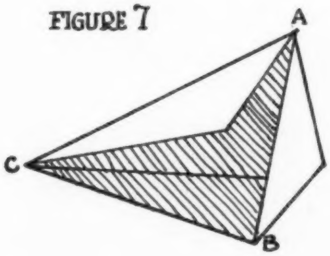



FIGURE 6

*Use*: To teach how to hit a bowl at billiards by reflection.

Suppose one bowl to be at point *A*, and that which you would hit at the point *B*, and *CD* the billiards table. Imagine then a perpendicular *BDE*, and take the line *DE* equal to *BD*. If you direct the bowl from point *A* to *E*, the reflection will carry it to *B*. For in triangles *BFD* and *EFD*, the side *FD* being

FIGURE 7



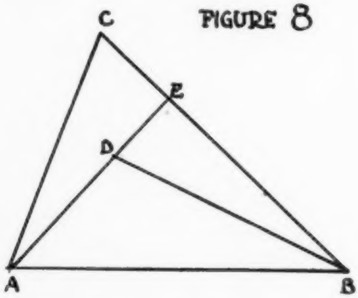

common and sides *BD* and *DE* equal; and angles *BDF* and *FDE* right angles, the angles *BFD* and *EFD* must be equal. But angle *ACF* equals angle *EFD*. Therefore, angle *ACF*, the angle of incidence equals angle *DFB*, the angle of reflection.

3. If two angles of a triangle are equal the triangle will be isosceles.

*Use*: Thales used this proposition for measuring the height of obelisks by their shadows.

If you were to measure the height of the obelisk $AB$, observe when the sun is elevated 45 degrees above the horizon, that is to say, so the angle $ACB$ be 45 degrees; and by above proposition shadow $BC$, will be equal to height of obelisk $AB$. Since angle $ABC$ is a right angle, and the angle $ACB$ half a right

FIGURE 8




angle, the angle $CAB$ will be half a right angle. Therefore angles $BCA$ and $BAC$ are equal; and by above proposition, sides $AB$ and $BC$ are also equal.

The same height can be measured without making use of the shadow by taking a stand so far from the point $B$, as that angle $ACB$ may be half a right angle, which may be known by a quadrant.

4. If a small triangle be described within a greater upon the same base, the sides of the small one will be less than those of the greater, but they will form a greater angle.

*Use*: By above proposition we demonstrate in optics that base $AB$ viewed from point $C$ will appear less than when it is viewed from point $D$; according to principle that quantities viewed under a greater angle will appear greater. Therefore, Vetruvius advises not much to lessen the tops of very high pillars; because they being so remote from our sight, quickly appear slender enough without being diminished.

Our pupils are justified in constantly asking, "What is the *use* of geometry?" It is a more pertinent question even in

this highly utilitarian age than it was in 1818 when Craig felt that one of the chief reasons for bringing out a new edition of Euclid was the *need* for making clear to the pupils the *use* of the theorems of Euclid. Well may geometry teachers today go back to this edition of Euclid by Craig for material in helping students to answer their own question as to the use and application of geometry.

# THE ORIGIN OF OUR NUMERALS

By CHARLES POMEROY SHERMAN

Suggested, probably, by the present-day inquiry into the origin of things, it occurred to me to look up the origin of our so-called Arabic numerals; and I found, to my great surprise, from the authorities I was able to consult, that it is much disputed and is actually unknown.

This much I learned:

Before the end of the second century of our era numerals somewhat resembling those we now use were in use in India.

In the ninth century Abn Ja' far Mohammed ben Musa, surnamed Al-Khowarazmi, that is, a native of Khwarazm (Kiva), an Arabian mathematician, wrote a work on algebra in which he used for numbers the signs which he had obtained in India or Afghanistan.

In 1202 Leonardo of Pisa, Italy, translated, or paraphrased, that work into Latin, and thus introduced those numerical signs into Europe.

By 1400 those numerical signs in use in Europe were identical with those in use by us today; and they gradually supplanted, in general use, the clumsy Roman signs and the still more clumsy Greek.

Thus much for what we know; but we do not know who was the originator of those signs nor what was in his mind when he formed those seemingly arbitrary shapes.

Therefore, as we do not know—as we have no information whatever—it is allowable for us to conjecture what, probably, was then in his mind.

The savage looked at his hand, and saw four fingers and a thumb, five short straight lines; and thus he learned to count up to, perhaps, five. He looked at the other hand and saw five more short straight lines; and thus learned, less often, to count up to ten. In place of his fingers he used, to illustrate his numbers, small stones or short pieces of stick or short straight scratches in the sand or earth or on the rock.

The Roman also looked at his hand; but he saw also the V shape between the forefinger and the extended thumb, and conceived the idea, to save making five straight marks, to join

two straight lines at the end, thus imitating that V shape, and to make that V stand for the whole five straight lines.

One day, holding his two hands before him, palms outward, and happening to draw them together until the extended thumbs crossed, he conceived the further idea, to save making ten straight marks, or two V shapes, to cross two straight lines in the middle, thus imitating those crossed thumbs, and to make that X stand for the whole ten straight lines.

So the savage and the Roman used short straight lines, arbitrarily placed, for their numerical notation.

Was this same idea of *short straight lines arbitrarily placed* in the mind of the man who invented the beginnings of the signs which were found by Abn Ja' far Mohammed ben Musa to be current in the East?

Can we reduce our numerals to the requisite number of short straight lines without outraging probability? I think we can:

1 was, and is, one perpendicular line.

2 was evidently horizontal lines, one above the other.

3 was evidently three horizontal lines, one above the other.

4 was a square, four lines.

5 was a square, with a horizontal line added above (the beginning of another square); five lines.

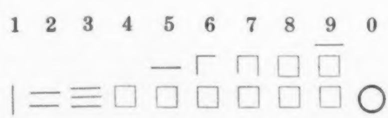
6 was a square, with a horizontal line above, to which a perpendicular line was added at the left end (the continuation of a square); six lines.

7 was a square with a horizontal line above, to which two perpendicular lines were added, one at each end (the continuation of a square); seven lines.

8 was two squares, one above the other; eight lines.

9 was two squares, one above the other, with a horizontal line above the upper square; nine lines.

0, a circle, no straight lines, nothing, was probably invented when tens were invented by setting a number to the left of another, so as to show that there was nothing added to the 1 standing to the left as ten.



You have before you our present numerals, and underneath each the shape in which it probably appeared to the mind of the inventor in the dim long ago.

You will notice that, after the first three numbers, the others are squares of four lines, upon which are gradually built up other squares.

Now how did those shapes become the shapes of our present numerals?

As time went on, writers tended more and more to substitute the easy curve for the difficult straight line, and not to lift the pen from the paper between detached lines but to join the two—which we will call *cursive writing.*

(How little the written letter or word of today resembles the accurate letter or word of print, shows the effect of cursive writing.)

The Roman numerals did not lend themselves to cursive writing, because they proceeded horizontally, but the Eastern numerals did, because they proceeded perpendicularly, and were written downward from the top, and therefore and thereby their aggregated lines assumed a single shape, the most convenient for use in mathematics.

Let us consider each number as changed from its original shape to its present shape by cursive writing:

1, as I have said, was and is the same.

2: the upper line was curved and joined to the lower line.

3: all three lines were curved and joined.

4: the right hand perpendicular line of the square was crossed over the lower horizontal line (thus making four lines) and the upper horizontal line was omitted.

5: the upper right hand and lower lines of the square were curved into a near-circle, and the left hand perpendicular line was omitted; a short down stroke of emphasis was prefixed to the beginning of the near-circle, and to this the upper detached line was added.

6: beginning at the right hand end of the upper detached line, a sweeping half-circle, with a concluding curve to the left, changed all the straight lines into one continuous curve.

7: the right hand line of the three upper detached lines was joined to the right hand perpendicular line of the square, and the other three lines of the square omitted.

8: the two squares were curved into circles and joined.

9: the upper square was curved into a circle, to which was joined the right hand perpendicular line of the lower square, and all the other lines omitted.

Thus, doubtless by slow degrees, the comparatively awkward combinations of straight lines and squares were ultimately condensed into shapes of flowing lines, easily written, each most obviously distinct from the others.

Thus did that unknown inventor, and the scribes succeeding him, give to the world a mathematical boon.

# LIVE PROBLEM MATERIAL IN ALGEBRA

By DWIGHT S. DAVIS
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Verbal problems in algebra have been, and, one might venture to say, still are the object of much destructive criticism from both students and teachers. By some inexperienced instructors, verbal problems are sometimes omitted from the outline of study. Other instructors often place undue emphasis upon problems and continue to drill on problems until the nerves of the parties of the first and second parts are exhausted. The teacher with a well balanced outline of work will use live verbal problems in driving home the principles of algebra. Verbal problems, then, should be a means to an end, not an end in themselves. They should assist in causing mathematics to function in solving real life situations.

We are all only too familiar with the hypothetical friends, *A* and *B* and their assistant *C* who are always involved in fantastic relationhsips as regards their finances, ages or occupations. With these and other equally suppositious sources of algebra problem material we are familiar. Familiarity breeds contempt and likewise lack of interest. Is it not time to retire these honored but time-worn friends, ere they earn the bitter reward of familiarity? Let us, then, muster to the service, young, vigorous assistants whom our students will find real playfellows rather than boresome abstractions.

But even the best of fellows must have an introduction before he will be admitted to the society of the classroom. Can we, the mathematics teachers, so introduce these assistants as to create and sustain the interest of our pupils?

The material which is about to be laid before you for your inspection and constructive criticism is presented only as a suggestion—a small germ from which, it is sincerely hoped, many new thoughts and ideas based on varied experiences may arise. An attempt has been made to include topics under one head or another with which students will find themselves acquainted, if not familiar.

Our first source of live material is found in that most ancient and fundamental science—agriculture. Here we may find

material for all phases of problems. To show the use of letters in deriving formulas for areas of different figures we have simply the abbreviation of our old arithmetical rules in the following: Area of a rectangle equals $ab$; area of triangle equals $\frac{1}{2}ab$. Such area problems are not new but the writer does not recollect having seen many problems like the following: In planting a large field of potatoes, the fertilizer indicator on the planter is set so that 1,500 pounds of fertilizer will be distributed to the acre. How much fertilizer is needed on a field of 40 rods wide and 64 rods long? Use your formula for area. From this it is an easy matter to have the student derive a formula of his own for finding how much fertilizer will be needed under varying conditions. If dimensions, for instance, are in rods the formula for area in terms of acres would be $\frac{lw}{160}$ and if 1,500 pounds of fertilizer are needed per acre he could derive this formula for the amount needed in terms of tons $\frac{3}{4} \cdot \frac{lw}{160}$ or $\frac{3lw}{640}$. Have the student make up formulæ for varying amounts of chemicals. From this one problem many others can be derived such as: (1) How much will it cost to ship that fertilizer from its place of manufacture to your home town freight depot? (2) What will be the cost of hauling it to the farm? (3) Weather conditions being average it is safe to assume we will get between 180 and 230 bushels per acre as a result of using high grade chemicals whereas the average yield used to be 45 bushels per acre. How much is the increase? (4) Figuring previous costs and final income what was the resulting net profit? Topic 3 may be made a source of material for comparative graphs.

Thus it is easily seen that one problem may be a veritable nest egg for a dozen others whose number and variety will vary directly with the ingenuity, general knowledge, energy, and spare time of the teacher.

In evaluation of formulas, agriculture furnishes us with a wealth of material. Let us for a moment glance at the spraying problem. The algebraic formula for a 5-5-50 Bordeaux mixture would be $10c$ equals amount, where $c$ equals 1 unit of chemical,

the two chemicals, lime and copper sulphate being in equal amounts. Example: We have a 100 gallon sprayer mounted on a two-wheel chassis. How many pounds of chemicals are needed in filling this sprayer? How many pounds of each?

Let us see to what the above will lead.

Suppose the force pump to the sprayer is geared to the wheels so that for every revolution of the wheels, the pump goes through two complete cycles. In one cycle it pumps one pint. How many feet can the sprayer be run before going dry, if the wheels are five feet in diameter? In this problem we use our formula for the circumference of a circle, $2\pi r$. After a few experiments the teacher can have the students evolve a formula for the distance which that particular sprayer can travel without refilling. He would find this to be in this case $800\pi r$.

For drill in problems where the student makes his own equation, the mixing of chemicals and feeds furnishes bountiful material. The following will serve as an illustration.

In a standard corn fertilizer there are five times as many pounds of acid phosphate and six times as many pounds of cotton seed meal as there are pounds of kainit. There are 80 pounds of "filler" to the ton. How many pounds of each ingredient must be bought if we wish to mix a ton?

I should say in passing that I am purposely omitting all explanation of technique and technical terms, knowing that the resourceful teacher will readily familiarize himself or herself with them as the need arises. Before launching the student upon these problems, the teacher should make easy the visualization of the problem. Constructive imagination is one of the most potent factors in intellectual development.

Statistical graphs find use in observing crop yield from year to year, increase of products with improved tillage, profits, law of diminishing returns, etc. These figures as well as other valuable data may be found in the year-book of the Department of Agriculture.

Areas of fields furnish material for quadratic problems. For example: "A rectangular strawberry bed covers $3\frac{3}{4}$ acres. The length is 10 rods greater than the width, what are the dimensions of the field." Here again we evaluate the formula, namely $A$ equals $lw$, but we must translate one of the two letters $l$ or $w$ into the terms of other, such as $l$ equals $w$ plus 10, or $w$ equals

$l$ minus 10, it makes little difference as long as the mind of the student is clear upon the subject.

One finds simple machines like levers and pulleys in daily use by those engaged in agricultural pursuits. How many farmer boys plod along behind the plow in the spring, carrying a crow-bar with which they "pry" uncovered stones from their lodgings! And how often they wonder as they work away, just how many pounds they are really lifting! For instance, ask the boys this question and ask for a diagram. "You are at one end of a 5-foot crow-bar and a 100 pound rock at the other. If you get your purchase (fulcrum) one foot from the rock, how much must you bear down to stir it?" This should be correlated with instruction in proportion. Other problems in levers may deal with ways of hitching teams, carrying burdens on a pole, weighing by steelyards, making deadfalls and the like.

Pulley problems may be made up to deal with unloading hay, by means of the horsefork, dressing carcasses, lifting motors, etc. Here again we have evaluaton of formula: $F$ equals $\frac{W}{N}$.

At just about this time the question arises as to how far we may stray into separate fields without encroaching upon other subjects or neglecting algebra. The answer to the first part of the question is given by more than one science teacher when he exclaims, "How much more time I could spend teaching real science if my students actually knew how to use formulas!" The answer to the second part of the question each teacher must answer for himself, for there is as yet no known method of standardizing mathematics teachers as one may standardize parts of a great machine.

The next main topic which we may find fruitful is the field of modern mechanics. Let us first turn our attention to the general subject of motors. We usually think of motors in terms of horsepower—a unit which to you and me immediately translates itself to mean 33,000 foot pounds per minute but which to the average student usually means just what the term implies—the power of one horse. Since pressure of some sort is needed in all engines, a few words along this line would be well in introducing the subject of horsepower formula. This formula is one of the most easily remembered of any: $\frac{Plan}{33{,}000}$ A few introductory

problems, such as the following should be solved before entering into more difficult situations. For instance, "The mean effective pressure on a piston of a Mallet locomotive is 150 pounds per square inch, the area of piston-head is 48 square inches, length of stroke $1\frac{1}{2}$ feet, number of single strokes per minute 120. What is the horsepower of the locomotive at this speed?"

From examples of the above type we may safely proceed to leave out one of the variables, or even two, as we come to equations containing two unknowns. Equations involving such formulas should be used only in correlation with advanced mathematics since factors enter which are beyond the experience and interest of the elementary student. Before we pass on, we might cite a problem of this nature: "A Mogul locomotive with cylinders of 18 inch bore, 26 inch stroke and a mean effective pressure of 60 pounds per square inch is designed to run at 60 miles per hour under above pressure. What is the horsepower and number of strokes per minute if the drivers are 6 feet in diameter? Or, what horsepower does it develop at that speed?"

From steam motor power we naturally come to internal combustion motors—the best known of which is the gasoline motor. From any automobile manufacturer, one may obtain circulars containing specifications which may be made use of in many problems of interest to those studying advanced mathematics.

So far most of our discussion has been upon subjects likely to be of more interest to boys than girls. Domestic science should furnish a multitude of problems also. For example, we can find use for our formula $a$ equals $lw$ in problems involving plastering, papering, carpeting, and painting. In a problem of papering a room, negative numbers may be used to denote areas which will need no paper—windows and doors.

Or in a ventilation problem one may use simultaneous equations. "Two rooms together contain 2,600 cubic feet of space, one being 200 cubic feet less in space than the other. What are the capacities of the room?" This, of course, may be done in an equation of one unknown but with more steps involved.

In ventilation we also have evaluation of formula. Since air should be changed at least three times per hour we can find the amount of air per person by this formula: $\frac{3hlw}{P}$. ($P$ being the

number of persons occupying the room.) This may be used thus: Three people sit in a room in which the air is changed three times per hour, dimensions are 10 feet by 15 feet by 9 feet. How much air does each person receive per hour? These problems may be altered by omitting any one of the four variables in the formula. Example: A recipe for molasses cookies calls for three cupfuls of ingredients and a teaspoonful of soda and spices. Make out the recipe if there are equal amounts of sugar and butter, one-half as much molasses as sugar and half as much boiling water as sugar, all measured by the cup.

Let $x$ equal sugar and butter
$\frac{1}{2}x$ equal molasses and boiling water,
etc.

Recipes are an almost inexhaustible source of problem material. Have girls make a recipe book along with the algebra.

Household budgets one finds of inestimable importance. The division of income may be graphically illustrated by block diagrams or circular plotting. The former will probably take less time and prove of as much value. One should not, in interest of objective work forget the valuable subjective matter contained in this subject. For instance: A monthly income of $200 is divided so that twice as much is spent for heat, light and water as for church, twice as much for the bank for interest on mortgage as for heat, water and light and four times as much for food as for the interest on the mortgage. If $115 is spent in this manner, how much remains to be spent on each of the other items?

Banking is a subject with which we are all interested and already much material is made up with the formula $A$ equals $PRT$ as a basis. However, we might with profit obtain data from a cooperative bank and make a graphic study of the growth in amount if one dollar is deposited monthly for the given period of years. Example: Plot a graph showing the growth of value of a five share holding in the . . . Cooperative Bank from the first deposit to maturity. Use intervals of six months. Reliable stocks and bonds furnish an interesting study. Students may pretend to buy a certain listed stock and keep a daily graph of the standing of their respective holdings. As an object lesson it might be well to pick out some "wild-cat" stock and have graphs

made. The contrast between amount of fluctuation is often striking.

Athletics hold forth an appeal to all members of the class. In the fall football scores may be made the basis of problems; especially will it aid in livening up Monday morning classes to pick scores of games which will lend themselves readily to manipulation. This same will hold true in all sports throughout the year. In baseball we are able to find material for decimal work in figuring up batting and fielding averages of players, team standings, games to be won if standing is to be maintained, strike outs, and assists by pitchers, etc. A few typical problems of each sport follow.

In Saturday's game the Bears outrushed the Bulldogs by 18 yards. In all the ball was rushed 314 yards. How far did each team carry it?

In a recent cross-country run Tom Brown won the event with Whitney coming in second. Their running times added together give 86 minutes; twice Brown's time added to half Whitney's time is 106 minutes. What was Brown's time over the six-mile course?

In last night's basketball game, Seaver made six times as many points from shots from the floor as from free tries. How many shots did he sink from free tries if his total score was 21?

The hockey team has scheduled 18 games. Of these it has so far won eight and lost three. How many more must it win to have a standing of approximately 77 per cent?

In the "Home Run Club" Williams has three more than five times as many as Hauser, two less than twice as many as Miller and 12 less than four times as many as Meusel. Together they have 58. How many has each?

If we added the standings of the two Boston clubs, the combined standing would be .43 less than the Giants who have won .598 of the games they have played. If the Red Sox are .047 higher than the Braves, what is the standing of each?

All racing events, motor cars, yachts ,motor boats, horse races, foot races, track events may be used by keeping clippings made of times and records.

"In the Indianapolis Sweepstakes Jimmie Murphy an American driver of an American car, won the 500-mile event in record time. His time and the rate per hour of Hartz, who came second,

totaled 188.01 miles. If Murphy beat Hartz by .95, what was his rate per hour?"

The field of aviation contains ample material for problems. If one wishes material for statistical graphs or corresponding graphs many up-to-date publications will give specifications of various types of machines. The post office department will furnish data regarding aerial mail progress. A typical problem might be in positive and negative numbers such as: By what signs would you denote the following items relative to an H-16 bombing plane: four 250 pound bombs, six passengers totalling 900 pounds weight, machine guns and ammunition totalling 150 pounds, tonnage and fuel weighing 1000 pounds, weight of body and accessories 3 tons, lifting torque of 12,000 pounds?

Or in simultaneous equations: A navy F-5-L seaplane will make 70 knots against a wind but flying with a wind whose velocity is half again as great it will make 100 knots. What is its ground speed on a calm day and what was the speed of the wind?"

Furthermore we can calculate gliding ratio in regard to altitude, speed as varying with wind and revolutions per minute of motors and horsepower of motors. Here is a remarkable chance to introduce work in force diagrams, compass points, elementary navigation problems concerning points with which the students are familiar. If a course and distance indicator can be obtained it would vividly show the benefits of a mathematical training.

In regard to radio, one hardly ventures to say from one day to the next what is fact and what is fiction. Suffice it to say that one who regularly reads a newspaper, *The Literary Digest*, and a scientific magazine will have no difficulty in making up problems. For example in a recent issue of *The American Magazine*, Bruce Barton stated that when there is no more than ten per cent difference in wave length of nearby sending stations, interference will result. In tonight's paper you will read that two stations are to send at certain wave lengths. The sum of the wave lengths is 810; if the lesser were doubled and the greater diminishd by two-thirds, their total will be 1,320. Will they interfere? This problem will lead to others.

Under building and construction we have an opportunity to study girders, supports, torque and strain and the like. We may study composition of materials with the appertaining formulæ.

Under this heading, too, we may begin to supply numerical trigonometry problems.

Our post office department, being the most highly organized and developed business in the world, furnishes us with material for problems. From any convenient post office we may obtain mail rates for varying classes of matter from which problems of progressing difficulty may be devised. Aerial mail statistics make a very interesting study when we consider rates and time saved.

Marine problems may be presented to a class if the subject proves interesting. Problems involving boats of types from the diminutive cat-boat to the trans-Atlantic liners may be devised. Graphs can be made showing development of speed in crossing the Atlantic, growth in individual tonnage of ships and respective standing of varying countries in marine business.

Surely, then, with such means at our disposal we can, each in his own manner, apply verbal problems in algebra to those phases of our modern world which seem to hold forth the greater appeal to our students.

### Summary Outline

I. General Statement.
- A. Historic nature of problems.
- B. Present nature of problems.
  - 1. Transitory period in which texts are either,
    - a. Conservative, i. e., adhering to old type problems.
    - b. Progressive, i. e., presenting problems of every day interest.
- C. Tendencies in presenting verbal problems.
- D. Need of new material.

II. Presentation of Verbal Problems.
- A. Problems must be couched in every day English.
- B. The instructor as a salesman must secure from pupils,
  - 1. Favorable and undivided attention.

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AUTHOR'S NOTE: The author realizes only too keenly that his presentation has only scratched the surface of the fertile field of possibilities, but limits of time and space make a complete exposition of the previously enumerated sources undesirable. If, however, this introductory paper meets with approval, it is the author's intention to make a complete project of each of his main divisions of material.

2. Interest, which may be secured by
   a. Sheer absurdity of problem or
   b. Application of problem to every day things of interest to student.
3. Desire to
   a. Find how problem will work out and
4. Decision and action,
   a. Which, if interest is to be maintained, must be followed by
5. Satisfaction with results.

III. Problems must be from sources of familiar interest to the pupil who is interested in
  A. What he has done.
  B. What he has seen.
  C. What he has read of his own accord.

IV. Division of Material.
  A. Agricultural under which may be found
    1. Formulæ problems in area.
    2. Distribution over area.
    3. Formulæ for sprays, feeds and fertilizers.
    4. Statistical graphs on
      a. Yearly production.
      b. Profits.
      c. Weather.
    5. Fencing (diagrams, test of logical reasoning power)
    6. Leverage.
  B. Mechanical.
    1. Formulæ in steam and gasoline engines.
    2. Variables.
      a. That is, variability of horsepower with $P$, $A$, $L$ or $N$; variation of speed with $2\pi r$; variance of gas consumed with horsepower.
    3. Regular linear equations drawn from catalogues, booklets, etc.
    4. Percentage, friction, efficiency.
  C. Domestic Science.
    1. Area problems,
      a. Carpeting, papering, plastering and painting.
    2. Temperatures.

3. Diagramatic study of budget.
4. Recipe problems.

D. Banking.
1. Cooperative bank, graphs, percentage problems, etc.
2. Stocks and bonds—graphs.
3. Savings banks.

E. Athletics.
1. Baseball batting averages, fielding averages.
2. Football plays, positive and negative numbers.
3. Track records.
4. Motor racing (land and marine).
5. Swimming and diving.

F. Aviation.
1. Motors.
   a. Ratio of bore, stroke and revolutions per minute to horsepower.
   b. Lifting power varying with torque and horsepower.
2. Wing area.
   a. Functionality as regards sustaining power and speed.
3. Gliding angle.
4. Speed.
   a. Problems in driftage and head resistance.
5. Mileage.
   a. Endurance, gasoline, consumption.

G. Radio and electricity.
1. Evaluation of formulæ.
   a. Manipulation of $I$ equals $\frac{E}{R}$; Watt formula.
2. Wave length, sending radius.

H. Local distances and measurements.
1. R. F. D. routes.
2. Paper routes.
3. Computation of heights of buildings.

I. Building and construction.
1. Cost of materials.
2. Diagrammatic study of house lots and houses, buildings.
3. Tensile strength of girders; weight.

J. Post Office department.
1. Rates.
   a. Variability with different weights and zones.

2. Saving department.

K. Government.

1. Taxes.
2. Appropriations.
3. Population.
   a. Native and foreign-born.
4. Occupations.
5. Industries.

# MEASURING ACHIEVEMENT IN FIRST YEAR ALGEBRA

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While in theory the use of standard tests would seem to have as large a place in secondary school practice as elsewhere, in actual practice, their use has been relatively infrequent as compared with their use in the elementary school.

This fact may be accounted for variously. Among other possible factors operating to produce this situation may be mentioned the following:

1. Objectives in the high school subjects are not as clearly defined as in the subjects of the elementary school.

2. The content of the high school subjects is not as nearly standardized as is the content of elementary school subjects.

3. High school teachers have less professional training than elementary school teachers.

4. High school principals have less professional training than elementary school principals and their responsibilities as supervisors or superintendents of their school have not yet been nearly as well developed.

5. Satisfactory tests for high school subjects have not been forthcoming.

As regards the subject of algebra it can hardly be said that objectives and content are not as clearly defined and standardized as the average elementary school subject. It is quite true that the high school teacher of mathematics has not as much professional training as the normal school graduate teaching in the elementary school. On the other hand his mathematical training and ability tends to offset this lack through the predisposition they provide him in the direction of statistics and educational measurements. It is true that high school principals have not encouraged the use of standard tests in their school as much as have principals of elementary schools.

The author of this article believes that the chief reason that standard tests and scales have not found a large place in technique of the instruction of secondary school algebra is due not so much to the first four of the five factors listed in the preceding page as to the *lack of satisfactory tests* for measuring

achievement in algebra. If high school teachers of mathematics could be brought to feel that standard tests or scales for algebra actually measured achievement of the objectives towards which they were working, no further encouragement in the use of such tests and scales would be necessary to insure their widespread use.

The author has believed that the following features of existing tests for algebra have served to discourage their widespread use.

1. *They do not contain exercises of sufficiently wide range of difficulty.* An adequate test must afford a measure of varying degrees of power as judge by ability to work exercises of varying degrees of difficulty.

2. *They do not measure the various types of problems included in the topics for which they are intended.* Very often all exercises of a test, say for division are all of one, or two, or three types, while there are as many abilities as there are types which in division as in other topics runs well over several times two or three.

3. Some contain exercises which ordinary first year pupils cannot work, thus *reducing the number of exercises by which the pupil is actually tested to a number altogether too small.*

4. Some tests provide for two scores—rate and accuracy. In life situations requiring the use of algebra, the importance of speed as compared with the importance of accuracy is inconsequential. *Any attempt to measure speed simultaneously tends to affect the measurement of accuracy.* For accuracy, the pupil must work at his optimum speed, not his maximum speed.

5. Certain of the *tests do not sample the field of algebra adequately* and hence do not afford an adequate instrument of measurement.

6. Certain tests *do not furnish opportunity for diagnosis.* All exercises of the test are of equal difficulty and of the same type.

7. In certain of the tests *it is impossible to segregate the measures of different abilities.* A large number of abilities are involved in one test.

With the above considerations in mind, the author has attempted to devise and standardize two series of tests to measure achievement in first year algebra and to afford a rough means of diagnosis.

In the spring of 1921 there was published Series A of the Douglass Algebra Tests. Series A consisted of four tests of ten exercises each as follows:

Test I—Addition and Subtraction.
Test II—Multiplication.
Test III—Division.
Test IV—Solution of simple equations.[1]

The exercises were selected to cover the various subtypes of problems under each of the four topics and included exercises ranging from very little to a great deal of difficulty.

The exercises were weighted in terms of median deviation (P. E.) on the basis of the percentage of pupils solving each. Norms or median scores were calculated from test papers written by 938 pupils in fourteen high schools distributed among five Middle West and Northwest States. Several thousand of this original Series A were sold and used. The norms seemed to be quite adequate but it became apparent that the tests were inadequate in two respects:

1. There was a need for an alternative form.

2. The tests (for the fundamentals) did not cover the field of the first year algebra.

Further experimentation revealed the desirability of discontinuing the practice of weighting the exercises according to their difficulty. Correlations showed very little inaccuracy in measurement resulting from this procedure. There were some criticisms too of the exercises used in the original tests.

As a consequence of these things, a new Series A was devised—Form II. This series was in most respects exactly like its predecessor. It was found, however, that it could not be used as an alternative form. Changes in certain of the test exercises precluded that. New norms were determined from 748 test papers well distributed geographically and among small high schools and large ones. In this series no weights were used. A third series, containing exercises corresponding in type and difficulty to the corresponding ones in Series A, Form II, was devised. This alternative series replaces the original Series A, as Form I.

[1] For a brief account of the derivation and standardization of these tests see Journal of Educational Research, 2:396-404, (December, 1921).

These forms afforded, however, only a means of measuring achievement in the fundamentals of algebra, addition and subtraction, multiplication, division, and solution of simple equations. The author then set about to devise a second series of tests to measure abilities on other processes of first year algebra.

A communication was sent to a number of prominent teachers of mathematics in secondary schools, colleges, and universities, prominent for their interest in and contributions to secondary school mathematics, requesting those to whom it was sent to designate such topics which should be tested in first year algebra in addition to those tested in Series A. Below is a tabulation of the replies received:

| Process | No. of Mentions | Process | No. of Mentions |
|---|---|---|---|
| Simultaneous equations | 13 | Formulae and Substitution | 5 |
| Verbal problems | 12 | Fractional equations | 4 |
| Quadratic equations | 9 | Exponents | 3 |
| Fractions | 9 | Parentheses | 2 |
| Graphs | 9 | Mixed fundamentals | 2 |
| Radicals | 7 | Ratio and Proportion | 2 |

Several other processes received one mention each.

At about this time the preliminary report of the National Committee on Mathmatics Requirements was published. After noting the recommendations of this committee as to what algebra should be taught by the end of the ninth year and with the replies tabulated above in mind, it was decided to devise tests as follows:

Test I, Fractions.

Test II, Factoring.

Test III, Formulae and fractional equations.

Test IV, Simultaneous equations.

Test V, Graphs.

Test VI, Square root, exponents and radicals.

Test VII, Quadratic equations.

This arrangement of the tests seemed to fit best the order of topics followed by the majority of widely used algebra texts. Formulae and fractional equations are tested together, most formulæ are fractional equations. Because of a marked dissimilarity in the methods of solution, a separate test is made for simultaneous equations. Test V, Graphs, is intended to test reading and interpretation of and construction of the most com-

monly used types of graphs and the solution of simultaneous equations by graphs. Square root, radicals and exponents, are all related topics of involution and evolution depending on the laws of exponents.

No separate test for verbal problems was devised. Verbal problems are and should be taught in connection with the abstract processes involved in the problems and should be tested in connection with them. In accordance with this idea one verbal problem is included in each of Tests II, III, VI, and VII and two in each of Tests IV, V.

Quadratic equations taught in the first year of secondary school algebra are of the type $ax^2 + bx + c = 0$, where $ax^2 + bx + c$ may be factored and each factor set equal to 0, thereby determining the roots. The test exercises were chosen accordingly. These seven tests constitute Series B.

Norms for the tests of Series B were determined as for Series A. The number of papers used ranged from 315 in Test VI to 858 in Test III. Form II, an alternative form, was devised as was Form II of Series A.

Because of space limitations the tests cannot be reproduced here.[1]

To give an idea of the plan of construction of the tests one test of each form of each series is given.

Series A—Form I

Test IV—Simple Equations

Time—Nine Minutes

1. Solve for m:
   $4m = 12$
2. Solve for n:
   $4n = -16$
3. Solve for b:
   $2b + 2 = 12$
4. Solve for r:
   $6r = 7r + 3$
5. Solve for y:
6. Solve for q:
7. In the formula $s = bh$, solve for h, when $s = 36$, $b = 9$.
8. Solve for a:
   $a(2 - a) + 2 = 4a - a^2$
9. Solve for b:
   $$\frac{.2b}{5} = 4 - \frac{.3b}{5}$$

[1] Complete sample sets, including copies of both forms of both series and directions for giving and scoring, may be obtained from the Bureau of Educational Research, University of Oregon, Eugene, Oregon, at 20c each.

10. Solve for x:

$$x^3 + 5 + 3x + 2(x + 5) = 2x - x(2 - x^2) - 5.$$

Series A—Form II

Test I—Addition and Subtraction

Time—Seven Minutes

1. Add:

$$\begin{array}{r} 7m \\ 4m \\ \hline \end{array} = \text{Ans.}$$

2. Add:

$$\begin{array}{r} 5x \\ -3x \\ \hline \end{array} = \text{Ans.}$$

3. Add:

$$\begin{array}{r} 3x^2y \\ -7x^2y \\ x^2y \\ -2x^2y \\ \hline \end{array} = \text{Ans.}$$

4. Find the sum of:

$6a - 7x^2$, $3x^2 - 2a - 3y^2$, and $2x^2 - a - mn$

5. Simplify by collecting terms: $10a^2 + 5ab^3 - 6x^2y - 13a^2 - 4ab^3 - 2a^2 - x^2y - c^3$

6. Simplify by removing parenthesis and collecting terms:

$$2a - 3b^3 - (5a - 4b^3) - (4a + [a - b^3] - b^3)$$

7. Subtract:

$$\begin{array}{r} 7x \\ 4x \\ \hline \end{array} = \text{Ans.}$$

8. Subtract:

$$\begin{array}{r} -4m^2n \\ 2m^2n \\ \hline \end{array} = \text{Ans.}$$

9. Subtract:

$$\begin{array}{l} 8a^2 - 9x^2 + 7ax^2 \\ 4a^2 - 4x^2 - 3ax^2 \\ \hline \end{array} = \text{Ans.}$$

10. From $ax^2 + 7xy$ take $4ax^2 - 2xy$

Series B—Form II

Test I—Fractions

Time—Twelve Minutes

1. Find the L. C. D. (Lowest Common Denominator) of:

$$\frac{1}{x^2yz}, \quad \frac{1}{xy^2}, \quad \text{and} \quad \frac{1}{xyz^3}$$

2. Reduce to lowest terms:

$$\frac{48a^3b}{-96a^3b^3}$$

3. Change to fraction:

$$xy - \frac{x^2 + y^2 - z^2}{2xz}$$

4. Find the value of:

$$\frac{6b+c}{6b^2}-\frac{2c-1}{2bc}+\frac{b+c}{3c}$$

5. Simplify:

$$\frac{16xy^2}{27yz}\times\frac{9yz^2}{8x^2z}\div\frac{4xy^2}{6yz}$$

Series B—Form I

Test III—Formulae and Fractional Equations

Time—Fifteen Minutes

1. Using the formula $k=\frac{1}{2}bh$, solve for k when $b=4$ and $h=3$.

2. Express the following fact in a formula:

The interest due on a loan may be found by multiplying the principal by the rate of interest and the number of years the money is loaned.

3. Using the formula below, what does a equal in terms of b and c?

$$c=\frac{a+b}{2}$$

4. Following is the law of gear wheels, $\frac{t}{T}=\frac{N}{n}$, when t and T represent respectively the number of teeth in two gear wheels of different size and n and N represent respectively the number of revolutions per minute of each wheel.

If there are 60 teeth in one wheel and 20 in the other and the first is making 100 revolutions per minute, how many revolutions per minute is the second making?

5. Solve for r:

$$\frac{9r+5}{5r}-\frac{3r+1}{4r}=1$$

# ADVANTAGES OF A GENERAL COURSE IN MATHEMATICS FOR THE FIRST TWO YEARS IN HIGH SCHOOL[1]

By LOUIS A. McCOY
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Shall we have a general course in mathematics for the first two years in high school, or shall we stick to the time-honored one year of algebra and one year of plane geometry? By the general course we mean a course, unified as far as possible from the standpoint of subject-matter, coherently connected, and consisting of some arithmetic, some algebra, some plane geometry, a little solid geometry, and the idea and the use of the function in numerical trigonometry. If there be any justification for such a course, it must be that it can do more for a pupil, give him better equipment, and more power, so that he can take his place as an intelligent member of the community if he should leave school, or be a greater aid to him should he continue his school work in preparation for college.

Let us consider first the advantages of such a course to the boy who leaves high school or at least quits mathematics at the end of his second year. What does he carry away with him that the other fellow didn't get? A good rational common-sense idea of our number system, the algebraic equation as a handy tool to use in connection with his arithmetic problems as well as his geometric formulas, both plane and solid, the power to think clearly, exactly, and logically, with the ability to "carry over" this power into other activities as a result of numerous applications, a better and more concise knowledge of spatial relations, the ability to use the graph, some information of the important idea of similarity, and the very helpful notion of the function in numerical trigonometry.

Of course we dont' have the boy with us very long. There is an ice cream firm which very cleverly advertises, "We can't make all the ice cream in the world so we make the best." Now, we teachers can't give the boy (we use the term "boy" in the common gender) all the mathematics that we would like to, therefore we must give him the best. Arithmetic has some "best," algebra has some "best," plane geometry has some

[1] Delivered before the Association of Teachers of Mathematics in New England, December 9, 1922.

"best," solid geometry has some "good," quite a lot of bad, and Book VI, and there is much "good" in numerical trigonometry; so let's pick out all these "best" parts of each subject, wield them into a unified whole, and give it to the student while we have him in our clutches. "Can he take them together?" you may ask. That is just the way he'll have to take them. What other way is he going to get them, after he leaves us? Is somebody going to come up to him and say, "Now I want you to do this, this is algebra, or this is part of geometry"? All he will be told is, "Here it is, go to it." The general way is surely the natural way; let us teach the youngster the things the way he is going to use them. As nearly as we can, have them function for him in school the way they will function for him afterwards. we could hardly blame a healthy, robust youth if he picked up a book and saw a whole flock of "$x$'s" and "$y$'s" and complex fractions, and pairs of simultaneous quadratic equations, and indicies and radicals and the like, if he should gingerly close the book and remark, "So this is algebra." If we would arouse his interest, he must see something that he can soon use or at least fit into his own experience.

The general course gives the student a much more intelligent viewpoint of the relation between the various subjects, especially so since he isn't supposed to know that he is getting parts of different subjects. When a young boy eats a pretty tasty salad, he doesn't do much worrying about the ingredients which go to make it up. He is using a geometric fact to give him an algebraic equation, he is using algebraic equations to establish geometry facts and to solve numerical exercises from geometrical formulas; there is a continual intertwining, and above all he is being pulled away from the purely mechanical rules of each subject, and looking facts squarely in the face as facts to be used whenever the situation demands. He is becoming a thinker, not a manipulator. He is getting facts just about as he is going to meet them afterwards, and therefore it is quite natural and much easier for him to dovetail them into his previous experiences and hence obtain logical conclusions from given data.

The general course enables the teacher to get across the idea of application of principles already learned with much better success. All of us teachers know, that when we have taught a student an important fact or principle, and drilled him until he

has learned said fact or principle, then there is one step further to go. We must have him use the fact or principle at least once or twice while he is with us, or invariably he doesn't see that it is of any particular use when the situation calling for it arises. We give a boy a hammer and show him a nail, and then we most certainly must watch him use said hammer on said nail, before we are absolutely sure that he realizes that the hammer will drive the nail. In the general course, if we equip a student with the algebraic tool, equation, we can watch him use said tool to establish a geometric fact. Conversely, if we have proved with him a geometric fact, we can watch him get the algebraic tool, equation, and then continue to use the tool to find the missing dimension or answer the given question. He appreciates the formula and the equation because he sees them work. There are plenty of facts in algebra and plane geometry that are generally considered more difficult and rather disconnected; then these are the ones to be left out of our general course. On the other hand there are some facts of arithmetic, algebra, and geometry which admit of much readier application than others, and these are just the ones we want in our general course. Let's not worry too much about the amount of mathematical technique the youngster is getting at this stage. In the main, we are not dealing just now with the boy who will have very much need for mathematical technique. We want to make this fellow an intelligent citizen, and give him the best mathematics available to help him become one.

The general course can give the boy or girl some drill in logical demonstration and reasoning without piling on one hundred and fifty theorems to do it. He can be shown how to attack a geometric proof just as he would attack a question to be debated by explaining clearly the terms in the hypothesis, by telling exactly just what fact he is going to try to establish, and then proceeding sensibly and logically to prove it. This faculty of exact thought should be the proud possession of our citizenship; too many that don't have it solve our economic and social problems for us. Here's the call for some, not much, demonstrative geometry.

Now for the fellow who is fortunate enough to stay in high school and continue his mathematics in preparation for college. Will the general course for the first two years handicap

him so that he can't get enough algebra, and plane geometry, and solid geometry, and trigonometry so that he can get ready for his college examinations the next two years? On the contrary, it is a distinct asset even to him. When he tackles his college preparatory mathematics at the beginning of his third year, he goes at it with his head up. New situations don't trouble him as much as they otherwise would. He has become more self-reliant; he'll say to his pal next to him, "Let's wiggle it out." He sees indicated operations much more clearly; he can read the mathematical language. It is true that he hasn't had very much practice in complicated manipulating; he hasn't mastered very much technique; but it doesn't take long for him to get it, and he gets it much more intelligently. He can still keep going if he gets off the beaten path. He has had the simpler principles of algebra, geometry, and trigonometry; he has met them with direct applications and he can use them as facts in helping him master the more complex principles of the high school mathematics. We'll have to grant that the third year is a busy one, but the fourth year is a real pleasure.

The general course is no longer an experiment; it is here. So we might look at a few facts that have already happened. In our large general high school where mathematics is purely an elective, about five hundred fifty elect it the first year, the same number the second year, three hundred fifty the third and about two hundred the fourth. The enrollment of the school ranges from twenty-five hundred to three thousand boys. In the light of these numbers we can hardly say that the general course for the first two years kills interest in the subject. Almost half of our senior class elects fourth year mathematics and less than half of these are aiming for college. More than two-thirds of our junior class elect M3, mostly plane and solid geometry, and when that number of students elect solid geometry, something a little bit different has happened in their study of plane geometry. However, we do think, that the solid geometry in the M3 accounts for some of the mortality in the M4. Now then, if a general course in mathematics for the first two years in high school can give the student the practical values of the subject in such a way as to make him a thinker, and at the same time create in him the desire to come back for more mathematics, well, isn't that enough for the time being?

# ON THE PRECEDENCE OF NUMERICAL OPERATIONS

By ROBERT E. MORITZ
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"There are no parties among mathematicians, and hardly any disputes, because mathematicians have had the wisdom to accurately define the terms they use."—*Thomas Reid.*

"Mathematics has no means for expressing confused or obscure ideas."—*Fourier.*

(1) "In a polynomial each term is to be treated as if it were enclosed in parenthesis . . . Otherwise all operations of addition, subtraction, multiplication, and division are to be performed in the order in which they occur."—*Schultze, Advanced Algebra.*

(2) "In a series of operations involving addition, subtraction, multiplication, and division, first the multiplications and divisions shall be performed in the order in which they occur. Then the additions and subtractions shall be performed in the order in which they occur or in any other order."—*Hawkes, Luby, Touton, Second Course in Algebra.*

(3) "In any chain of multiplications and divisions the order of the constituents is indifferent, provided the proper sign be attached to each constituent and move with it."—*Chrystal, An Elementary Textbook of Algebra.*

(4) "To prevent mistakes, and to make usage uniform, the following rules have been adopted: In an expression, etc.,

"All multiplications are performed first, and these may be taken in any order.

"All divisions are performed next, and these are taken in the order in which they occur from left to right.

"Finally, additions and subtractions are performed, and these may be performed in any order."—*Slaught and Lennes, Elementary Algebra.*

(5) "In an expression like $6 \div 4 \times 3 \div 4 \div 2 \times 3$,

"We must not divide 6 by 4 until 4 is multiplied by 3 nor divide the 3 by 4 until the 4 is divided by 2 nor (must?) the 2 (be?) used as a divisor until it is multiplied by 3."—*W. D. Henkle* (Joint author of Stoddard and Henkle's University Algebra), *Educational Notes and Queries,* Vol. 5, p. 36.

The foregoing rules are not relics of a dim and distant past but they are, with one exception, quotations from highly popular current textbooks. The last is from a writer known and honored by many of the older generation of present day mathematicians.

We have here a fair approach to an exhaustive enumeration of possibilities. Let the student operate as his fancy dictates, following any order or no order, and he will find some authority to support him. In particular he is authorized to perform:

(1) Multiplications and divisions as well as additions and subtractions in the order in which they occur.

(2) Multiplications and divisions in the order in which they occur, but additions and subtractions in any order.

(3) Multiplications and divisions in any order.

(4) Multiplications first in any order, then divisions in the order in which they occur.

(5) Multiplications and divisions in the reverse order from which they occur.

With textbook writers at sixes and sevens one naturally expects a similar state of confusion among teachers and students. However, this would not preclude some prevailing usage. In order to discover such prevailing usage the writer recently put the question $12 \div 3 \times 4 = ?$ before eight different classes in freshman mathematics. The combined number of students questioned was 162. Of these 72 gave 16 as the answer to the question asked them, 72 gave 1 as the answer, and 18 expressed themselves in doubt. Moreover, this even division was true not only for the aggregate number of those questioned but it characterized the count in seven of the eight individual classes. Of the eighth class six gave 16 as the result, sixteen gave 1 as the result, and seven were in doubt.

This indicates a deplorable condition of affairs, a state of confusion for which a remedy should be found if possible. What is the reason that mathematicians who have for so long been reputed "of submitting nothing to authority" and "of publishing only what they know to be true and can make good by invincible arguments" and "of preferring to be silent, choosing rather to acknowledge their ignorance, than to affirm anything rashly" should have been caught in a quagmire at the very threshhold of their science?

It is the firm conviction of the writer that the cause of the confusion lies in the tacit assumption that the order in which multiplications and divisions are to be performed is a matter of convention and that, inasmuch as there is no established convention recognized by all, each author is free to choose the rule that suits him best.

The remedy consists in showing that the aforementioned assumption is unwarranted. All arithmetical operations are interrelated in such a way that when a convention relating to the sequence of any one of these operations has once been agreed upon, no further choice is possible. Consistency demands one rule of precedence and denies all other rules. The *principle of no exception* is operative here as elsewhere.

There is a universal agreement in interpreting $M - a + b$ to mean that $M$ is to be diminished by $a$ and the result increased by $b$. Consistency demands that $M \div a \times b$ shall mean that $M$ is to be divided by $a$ and the result multiplied by $b$.

To be consistent, those who teach that $M \div a \times b$ means $M \div (a \times b)$ should likewise teach that $M - a + b$ means $M - (a + b)$.

Likewise there is a universal agreement in interpreting $M - a - b$ to mean that $M$ is to be successively diminished by $a$ and $b$. Consistency demands that $M \div a \div b$ shall mean that $M$ is to be successively divided by $a$ and $b$.

To be consistent, those who teach that $M \div a \times b$ means $M \div (a \times b)$ should likewise teach that $M \div a \div b$ means $M \div (a \div b)$ and that $M - a - b = M - (a - b)$.

The universally accepted interpretations of $-a + b$ and $-a - b$ enable us to write

$$+a + b = +b + a, \quad +a - b = -b + a, \quad -a - b = -b - a,$$

and with the corresponding interpretations given to $\div a \times b$ and $\div a \div b$

$$\times a \times b = \times b \times a, \qquad \times a \div b = \div b \times a,$$

$$\div a \div b = \div b \div a.$$

Like relations hold for expressions involving any number of constituents. The commutative law holds for constituents affected with the sign $-$ and $\div$ as well as for constituents affected with the signs $+$ and $\times$. In fact, there is now a complete parallelism between formulas involving only additions and

subtractions and formulas involving only multiplications and divisions. Thus

$$
\begin{array}{ll}
+a+b=+(a+b), & \times a\times b=\times(a\times b),\\
+a-b=+(a-b), & \times a\div b=\times(a\div b),\\
-a+b=-(a-b), & \div a\times b=\div(a\div b),\\
-a-b=-(a+b), & \div a\div b=\div(a\times b),
\end{array}
$$

and

$$
\begin{array}{ll}
+(+a)=+a, & \times(\times a)=\times a,\\
+(-a)=-a, & \times(\div a)=\div a,\\
-(+a)=-a, & \div(\times a)=\div a,\\
-(-a)=+a, & \div(\div a)=\times a.
\end{array}
$$

All this is perfectly obvious and well known to students of such texts as Wheeler's *First Course in Algebra* and Chrystal's incomparable textbook. But why do so many of our textbook makers forget it?

The assertion has been made that consistency does not permit several interpretations of $a\div b\times c$ once the meaning or $a-b+c$ has been agreed on. Analogy suggests a single interpretation but analogy is not always a safe guide. What about proofs? Can you prove the commutative nature of divisions and multiplications from the commutative nature of additions and subtractions? C. Smith in his *Treatise on Algebra* proves that $a\times b\div c=a\div c\times b$, but he assumes that $a\div c\times c$ means $(a\div c)\times c$. He fails to show that any other assumption is inconsistent.

One natural approach will be through logarithms, which tie up multiplication and division with addition and subtraction by means of the law of exponents. Given

$$(1)\qquad a-b+c=a+c-b=a-(b-c),$$

then

$$(2)\qquad \log a-\log b+\log c=\log a+\log c-\log b$$
$$=\log a-(\log b-\log b),$$

and

$$(3)\qquad e^{\log a-\log b+\log c}=e^{\log a+\log c-\log b}=e^{\log a-(\log b-\log c)},$$

that is,

$$(4)\qquad e^{\log a} \div e^{\log b} \times e^{\log c} = e^{\log a} \times e^{\log c} \div e^{\log b} = e^{\log a} \div (e^{\log b} \div e^{\log c})$$

or

$$(5)\qquad a \div b \times c = a \times c \div b = a \div (b \div c).$$

The foregoing proof is unassailable if we accept the principle of no exception which in this case consists in taking the indicated operations in each of the equations (1), (2), (3), (4), (5) in the same order.

Another and better approach is through the concept of generalized arithmetical operations. It is possible to define a series or chain of operations containing an infinite number of links, such that addition and its inverse, multiplication and its inverse shall each constitute one of the links or rather double links. To show this let $a_n$ represent a staircase of $n$ $e$'s surmounted by $a$, thus

$$a_0 = a, \quad a_1 = e^a, \qquad a_2 = e^{e^a}, \text{ etc.,}$$

and $a_{-n}$ its inverse, thus

$$a_{-0} = a, \; a_{-1} = \log a, \; a_{-2} = \log \ \log a, \text{ etc.}$$

Now if we use the symbol $\underset{n}{+}$ to denote the operation defined by

$$a \underset{n}{+} b = (a_{-n} + b_{-n})_n,$$

and use $\underset{n}{-}$ to denote the inverse operation

$$a \underset{n}{-} b = (a_{-n} - b_{-n})_n,$$

we have a double infinite set of pairs of operations ($n$ may be taken negative as well as positive) of which any two consecutive pairs (pairs whose indices are two consecutive positive or negative integers) are related in every respect as are addition and its inverse, to multiplication and its inverse. When $n = 0$, we have the addition and subtraction processes, when $n = 1$, the multiplication and division processes, when $n = 2$, we have two new processes which are related to multiplication and division precisely as are the latter to addition and subtraction, etc.[1]

[1] The concept of a generalized operation of order n, addition and multiplication being of orders 0 and 1 respectively, is exceedingly helpful in disclosing certain hidden relations. For instance, it has long been held that there is no analog to the identity $b \times 0 = 0$ (See Chrystal, Algebra, Vol. 1, Chap. 1, 22). This conclusion is based on a fallacy. The general analog for $b \times 0 = 0$ is $b \underset{n+1}{+} (0)_n$ which for $n = 1$, gives $b \underset{2}{+} (0)_1 = b^0 = 1$.

The question now arises as to the meaning to be attached to a sequence such as $a \underset{n}{-} b \underset{n}{+} c$. If we disregard all existing conventions as to precedence we might choose with equal right either one of two rules,

$$\text{(A)} \qquad a \underset{n}{-} b \underset{n}{+} c = (a \underset{n}{-} b) \underset{n}{+} c,$$

or

$$\text{(B)} \qquad a \underset{n}{-} b \underset{n}{+} c = a \underset{n}{-} (b \underset{n}{+} c).$$

If we decide on the former we shall have for the cases $n = 0$, and $n = 1$, respectively,

$$\text{(A}'\text{)} \qquad \begin{aligned} a - b + c &= (a - b) + c, \\ a \div b \times c &= (a \div b) \times c, \end{aligned}$$

but if we accept the latter we have

$$\text{(B}'\text{)} \qquad \begin{aligned} a - b + c &= a - (b + c), \\ a \div b \times c &= a \div (b \times c). \end{aligned}$$

This shows that, disregarding all established usage, we might consistently choose either pair of rules (A′) or (B′), but we could not consistently choose the first rule of pair (A′) and the second rule of pair (B′). Now since the first rule of pair (A) is universally accepted, we must also accept the second rule of the same pair.

It remains to inquire what reason may have actuated textbook writers in attempting to establish a convention which, as has been shown, leads to inconsistencies. The answer is clearly stated in White's School Algebra (American Book Company, 1896) where after the statement that all multiplications must be performed before divisions, thus

$$\text{“}7 \times 8 \div 2 \times 7 \div 2 = (7 \times 8) \div (2 \times 7) \div 2 = 56 \div 14 \div 2 = 2,$$

and

$$a \times b \times c \div a \times b \div c = abc \div ab \div c = 1,\text{”}$$

we are told that "the above rule is based on the supposition that $b \times c$ and $bc$ have the same meaning." But why make a supposition which is neither necessary nor desirable? Why not say that $bc$ means $(b \times c)$ just as $b^2$ means $(b \times b)$? Accepting the implication that $bc$ is to be considered a single quantity the usual interpretation of such expressions as $a \div bc$ does not conflict with the interpretation of the symbols $\times$ and $\div$ which analogy suggests and consistency demands.

# THE PLACE OF THE CALCULUS IN THE TRAINING OF THE HIGH SCHOOL TEACHER

By BYRON COSBY
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"When the creation was new and all the stars shone in their first splendor, the gods held their assembly in the sky and sang 'Oh, the picture of perfection'! the joy unalloyed!

"But one cried of a sudden—'It seems that somewhere there is a break in the chain of light and one of the stars has been lost.'

"The golden string of their harp snapped, their song stopped, and they cried in dismay—'Yes, that lost star was the best, she was the glory of all heavens!'

"From that day the search is unceasing for her, and the cry goes on from one to the other that in her the world has lost its one joy!"—*Tagore.*

Sir Rabindranath Tagore, the Hindu poet, dreamer and writer, may not have thought seriously about the calculus when he sang the song just read, but to my mind the student who closes his course in mathematics without considerable work in the calculus, finds himself, in the words of Dr. Hillis, "like a lamp that is not yet lighted, a flower that has not yet blossomed and borne fruit, a newly made house across whose threshold the family has not yet come, a song unsung, a poem unread, a world incomplete." The lost star to some is wealth, fame, position but to the student it is Truth.

"Truth is life's factor and determinant
And we are workers in Truth's noble cause.
We yearn for Truth, we need its light: and Truth
Enters our Soul: it takes abode in us,
And consecrates our lives to higher service.
Not we own Truth, 'tis Truth that owneth us.

"Search for the Truth! Truth's problems are not vain.
Love thou the Truth! trust Truth, and live the Truth!
Walk on Truth's path and Truth will guide thee right."—*Carus.*

I think that while a great many of us believe that the search for Truth is the great thing we realize that we must recognize the demand for the practical or the applied in mathematics. The new textbooks, the magazine articles and monographs on mathematics keep rather pertinently before us the possibility of the use of mathematics in all fields of human activity. The student has a right to demand the chance to think. He should not be permitted to stop in his mathematical course until he has arrived

at the point where he has sufficient mathematical machinery to solve his problems. Books like *Shop Problems in Mathematics, A Course in Mathematics, The Mathematical Theory of Investment* and *Mathematics in Finance* are good types of books to demand of the student a greater knowledge of mathematics than he gets in the ordinary high school. With the present trend, and the fact that the calculus is the first place in the field of mathematics that we have methods for solving problems, is it not worth while to bring to the student in high school the best tools to work with? The simple notions of calculus may be easily used in high school, and being used make a demand that the high school teacher have a course in calculus. Our State Superintendent of Public Instruction requires only seven and a half semester hours of college credit in order to teach in the first and second class high schools in Missouri, which is an extremely low requirement.

When one is looking for reasons why a thing ought to be done, no matter how much of an innovator he may be, he looks to see what others have done and what they think ought to be done. I heard Dr. G. A. Miller read a paper before the Central Association of Science and Mathematics Teachers a few years ago, in which he quoted from the report of the International Commission on the Teaching of Mathematics, the list of studies that a student was advised to pursue if he expected to teach in the secondary schools, which list contained the calculus, differential equations, space analytic geometry, projective geometry, theory of equations, theory of functions, theory of curves and surfaces, theory of numbers and some group theory together with the following subjects among the list of applied subjects in mathematics; a strong course in mechanics, theoretical and practical astronomy, descriptive geometry, and some mathematical physics. The examinations in the Prussian schools include in addition to the high school subjects in mathematics, examinations in trigonometry with applications to mathematical geography, plane analytic geometry and the beginnings of differential and integral calculus. We find similar requirements in other European countries. At the Northeast Missouri State Teachers College we require of those who are to receive the mathematics diploma 22½ semester hours of college credit in mathematics, and 15 semester hours of college credit in physics.

The courses given in mathematics are selected from trigonometry, college algebra, surveying, analytic geometry, plane and space, history of mathematics, theory of equations and calculus. I am sure that we are not willing to lower our requirements and earnestly desire that as speedily as possible other schools meet the same requirements and that our State Superintendent of Public Instruction make it obligatory that all teachers of mathematics in Missouri high schools have more than the present $7\frac{1}{2}$ semester hours of college credit in mathematics in order to teach in the first and second class high schools.

But why study all of these subjects when we teach only the beginnings of algebra and geometry in our secondary or high schools. The answer is our problem.

First, it unifies all mathematics, it brings together the loose ends and gives the teacher or student unified or grouped wholes, and in doing so merely repeats human experiences. And, since our schools are to help the student to interpret and work out the useful experiences in life and these experiences appear in large groups or wholes, it is necessary for us as educators to present our work in such a manner, in order to have as little waste as possible. The greatest beauty in mathematics lies in its simplicity and economy. It is not economy when the student fails to see that the experiences in geometry are the same in different clothing as the principles in algebra, and it is not economy when we expect the student to work problems in a long and an ingenious manner in synthetic geometry when the primary principles of analytic geometry will give the solution immediately and clearly. Theer is no reason why part of the present work in algebra and geometry might not be omitted, and the primary and simple principles of analytic geometry and calculus substituted. I think part of the work on radicals, ratio, and proportion might be replaced by a discussion of the function from the standpoint of the calculus to the profit and pleasure of the student and teacher. All of the long and tedious work of calculation in building and loan, or life insurance problems in arithmetic might be worked more easily and more accurately as a rule, by the use of logarithms and the simple natural series represented by $e$. If we could present our work in such a manner that the student saw it as a complete whole, and was able to recognize the principle when it appeared even though in a different notation, it

would give him greater power, because he has less to remember and a greater chance on account of complete organization, to do more and better work in a more agreeable manner.

Second, the material and notation used in the calculus are more economical and more beautiful than the notation used in the more elementary work. And since we are looking for the simple, differentiated, beautiful and useful, even though the ideal be for use or the search for Truth is it not worth while to use that which seems to be the easiest to learn, to use and to remember.

Third, it is very doubtful if the ordinary high school student, and perhaps the average high school teacher of mathematics in Missouri, ever grasps the real reason why we are using graphic representation in the simple linear and second degree equations as the ordinary textbook presents the matter, and if the teacher has not had the work in the calculus she is not able to add very much to the textbook, nor give a very good explanation of the matter. The same is true of facts as velocity, acceleration and problems involving maxima and minima.

Fourth, if we are to introduce the principles as used in the college into our high schools, our high school teachers must be trained, or taught in terms of these subjects. And again, if our high school teachers are going to be honest, capable, earnest, aggressive and worth while should they not throw open the door to the child rather than try to teach through a mere crack in the wall? "When I was a child, I spake as a child; but when I became a man, I put away childish things. For now we see through a glass darkly; but then face to face: now I know in part; but then shall I know even as also I am known." (1 Corinthians 13:11-12.)

In our present tendency we are of the opinion that the work in our secondary schools, and even in our elementary school, should be in large units or blocks of work, and not widely separated, unrelated, unco-ordinated material as is found in our present-day arithmetics and algebras. Mathematics has been subject to attack, and will continue to be until we rearrange the material in such a manner as to eliminate the great amount of waste. The use of graphic representation so often left for the analytic geometry class, could be easily introduced in the high school algebra class by using the simple principles of the cal-

culus. The tangent slope at any point, and the maxima and minima can be found readily. In our work on the quadratic equations, the high school algebra method of finding the maxima and minima is slow and unwieldy, while the simple derivative gives the highest or lowest point promptly and without difficulty. Or the methods of the calculus will help promptly in determining the direction of the curve and whether or not it is increasing or decreasing at any point. Introduce the principle that the slope of the graph is shown by $dy/dx$, and that if:

a. $dy/dx$ is greater than zero, $y$ is increasing, and if:

b. $dy/dx$ is less than zero, $y$ is decreasing,

followed by the test for concavity downward and concavity upward, in the theorem:

In function $y = f(x)$, the curve is concave upward, when the value of $x$ as found in the first derivative and substituted in the second derivative gives a result greater than zero, and concave downward when the value of $x$ as found in the first derivative and substituted in the second derivative gives a value less than zero. The extent to which one wishes to carry the theorem, with any exceptions or discussions may be left to the capability of the teacher and the preparation and ability of the student. As an example, or exercise plot the curve represented by the equation

$$y = \tfrac{4}{3}x^{3/4}.$$

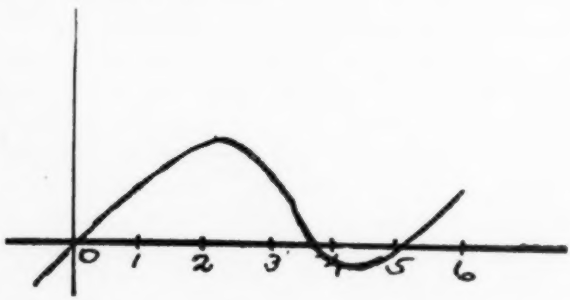
The first derivative gives the tangent slope at any desired point, and by choosing some definite point on the X-line, and using the test for maxima and minima, we find the curve approaches the line $x = {}^{25}/_{9}$ as an asymtote, and therefore the curve tends to cling towards the Y-line or axis. Or as another exercise, use the equation, $y = x^3 + 2x^2 - 13x + 10$, the first derivative of which gives when equated to zero the points at which maxima and minima appear, if any, in the theorem;

a. If $dy/dx$ equals zero for $X = x$, and the second derivative is less than zero at this point the function has a maxima for $X = x$;

b. If $dy/dx$ equals zero, for $X = x$, and the second derivative is greater than zero at this point the function has a minimum at the point $X = x$. The test for points of inflection or turning may be given. This may be done by putting the second derivative equal to zero and solving for $X$, and then if the third deriva-

tive is not equal to zero for $X = x$, the curve has a point of inflection at the point $X = x$. Some one may object that this is too much work and too complicated, but I shall answer that first, second and third derivative of all simple algebraic equations as ordinarily found are very simple, and probably no harder to remember than any other theorem in high school algebra, and if used at the time that the student is studying the problems of tangents in plane geometry, in the questions of belting, or in angles formed by a chord or a tangent, or by two tangents, and the discriminant in algebra, that $\underline{b^2 - 4ac = 0}$, and gives two coincident roots or in other words the condition of tangency, cannot help but give the boy a chance to see the unity in mathematics, and make him think the subject is worth while, especially if he has really tried to solve problems by the inefficient machinery of high school algebra. And most of us believe that the teacher who has this material will be more comfortable in trying to teach than the one who has not had an opportunity to look into the completeness of the topic.

A good problem in maxima and minima may be taken from algebra in the making of a box, from a rectangular piece of cardboard by cutting out squares from the corners of the piece of cardboard. If the dimensions of the piece of cardboard are 8″x10″, the volume of the box will be a function of $x$ in the notation of the following equation:




$$y = (8 - 2x)(10 - 2x)x, \text{ or } y = 4x^3 - 36x^2 + 80x.$$

To find the slope of the curve at any point find the derivative of the equation and substitute the value of the abscissa at that point. The first derivative is $y = 12x^2 - 72x + 80$, which van-

ishes for the values $x = \frac{9 + \sqrt{21}}{3}$ and $x = \frac{9 - \sqrt{21}}{3}$ and applying the test given above we find that one of these values gives the length of the square that is to be cut out of the piece of cardboard.

Velocity, acceleration and distance in the study of falling bodies growing out of the development of the work on a parabola was always more or less exacting to me, for a notation of expression. The equation of average velocity at which a point moved for a given length of time, is the distance divided by the time, is easily understood when expressed in the formula, but to see exactly what is meant by velocity at any particular moment of time with respect to the unit of measure of time was hard to see, until the problem was developed from the idea of the derivative. And again, it is not the average velocity that one is ordinarily interested in, but the actual velocity at a particular moment of time. A moving body may move fast or slow, gaining or losing at some particular moment of time. In the high school algebra we do not have given any method for finding the approximate speed at any moment of time, except by substituting $t - t_0$ for $t$, where $t_0$ is chosen relatively small, which method is more or less unsatisfactory on account of the fact that the substitution of a binomial is disturbing to the student of elementary algebra. The average velocity throughout the short period of time may be found rather accurately by making out a table and studying the question of limits, which question is rarely of any paricular help to the student in solving his problem. If the student attempts to make the table either for finding the average speed, or the slope of the tangent, he will find that it takes more time, patience and energy than the teacher or the pupil is willing to give and more time than the teacher can give, if he has to make preparation for tomorrow's lesson. And the results of the table are not sufficient accurate enough to be depended upon.

The problem becomes rather easy and most certainly short, when one can take the derivative or differential of the equation and immediately find the velocity at any particular period of time, or unit of time, by substituting in the derivative the value of the unit of time. It is easily illustrated in the equation $s = \frac{1}{2}gt^2$ whose first derivative gives the average velocity $gt$, or

32*t*. I am not sure that the teacher or high school student need have more than the working principle in mind, but I am quite sure that the principle as it may be explained is not harder than the types of problems given in some high school algebras. The problem of the derivative, or the differential is no more difficult when used in the elementary form and with simple algebraic equations than certain written problems, graphs, radicals or the principle of undetermined coefficients. The derivative lessens labor and makes the definition of velocity of a falling body at a given instant accurate and definite. The fundamental principle in mathematics calls for clearness and economy. And while the velocity can be shown readily on the graph of a parabola, it becomes more interesting when we take the second derivative of the space passed through by a falling body and find that this derivative, or the first derivative of the velocity is the acceleration of the falling body or moving point at the particular moment in question, which to my mind is one of the best ways to define acceleration.

The study of limits, finding areas of regular or irregular curves, or volumes of simple solids of revolutions, cannot but add interest and pleasure and help one to believe in himself, and work with a greater aggressiveness. Many more problems suggest themselves to one, and any one interested need not look for a list, but merely make a study of the principles that run continuously through mathematics and then minimize labor by introducing the higher principles whenever they are as easily understood, add greater clearness, give brevity and make one more comfortable in his thinking.

If one believes that the mathematician can get along with very few fundamental principles or facts, and we most certainly cannot give many in any one year, it is easy to convince him that if the few facts or principles are carefully developed it will be easy to get along with our work more economically, on account of the fact that the capable mind will be able to develop an infinite number of other related facts. And this is in keeping with the idea that a problem should be developed in a related co-ordinated manner, with unity and continuity, and makes for confidence and promise.

Other problems appear and demand attention, and they are of value and interest, but one cannot outline a course of study

in a short paper, and would not attempt it if possible, for no one's outline should be followed by the teacher who believes that the personality of the child must be preserved and educated, and also because this is not the problem, rather we are to give inspiration and hope, to put responsibility and demand, to give life and truth, to those who are to teach. One can hardly expect to teach a subject unless he has travelled further than the one he is to teach, for "Can the blind lead the blind? Shall they not both fall into the ditch?" (Luke 6:39.)

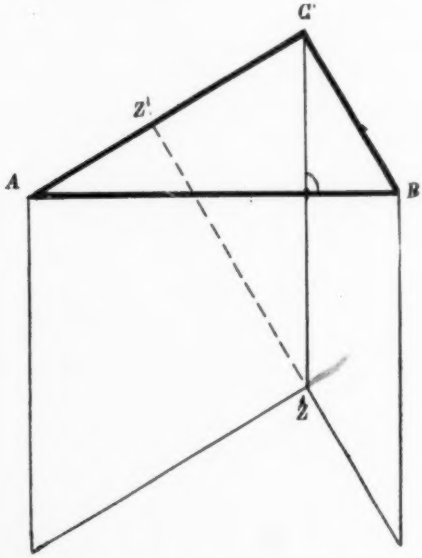
And then do we not all believe in evolution, that there is growth? "But I find that thy will knows no end in me. And when old words die on the tongue, new melodies break forth from the heart; and where the old tracks are lost, new country is revealed with its wonders." (Tagore.)

"In the meantime the air is filling with the perfume of promise." (Tagore.)

*A Proof of the Pythagorean Theorem.* In the triangle $ABC$, $a$ is perpendicular to $b$. Prove that $a^2 + b^2 = c^2$.

Draw $CZ$ perpendicular to $AB$ and equal to $AB$. From $Z$ draw $ZZ'$ perpendicular to $AC$ as at $Z'$. $ZZ'C \cong ABC$.

Call $a'$ the projection of $BC$ on $c$.




Call $b'$ the projection of $AC$ on $c$.

Complete the rhomboids $ZCB$ and $ZCA$.

The rhomboid $ZCB$ has two different area formulas; hence

$$a^2 = a'c.$$

Similarly, from $ZCA$:

$$b^2 = b'c$$

$$\text{then } a^2 + b^2 = c^2$$

Charleston High School, Boston. GEORGE W. EVANS.

*Reviews in Geometry.* It has been my experience that frequent and thorough reviews are very helpful in geometry. They serve to unify the subject matter for the student, make clear its

logical nature, and equip him with a fund of material for his use in the solution of problems. While the importance of well conducted reviews is obvious, the difficulties incident thereto are not few. They are likely to be either too fast for the poor student or too slow for the good one. In either case the result is unsatisfactory.

In an effort to get a satisfactory review it occurred to me that if it were possible to get slides and project them on a screen the problem might be solved. Accordingly I made drawings of the figures used in proving all fundamental theorems and had slides made from these drawings. At first I used a so-called "daylight" screen but upon experimenting I found that the slate blackboard made an even better screen. It is not necessary to have a darkened room for the figures are very clear and distinct in ordinary daylight.

By using these slides the figures are thrown on the blackboard, the student quotes the theorem suggested by the figure, and then the proof is given. Not all auxiliary lines need be in the figure projected on the board, in fact, I think there is a distinct advantage in not having them there, for the student will have to supply them otherwise. All such lines may be drawn on the blackboard in connection with the figure projected there. By this method it is possible to conduct a rapid and spirited review which I have found very satisfactory.

L. W. MURPHY

The Hotchkiss School.

*The Importance of Mathematics.*: A recent bulletin issued by The American Mathematical Society we find the following opening statement:

"Mathematics is one of the most vital intellectual interests of the human race. The extent to which a nation is concerned with this interest will always be a true index of the level of its civilization, because:

"(1) The Science which deals with number and space, with time, and energy, draws its inspiration from the deepest questionings of the human spirit, and busies itself with the most fundamental problems that challenge the mind.

"(2) Physical sciences are precise and unequivocal just in so far as they take mathematical form. Even social and biological sciences tend more and more to mathematical formulation and precision.

"(3) Man's mastery over the forces of nature and therefore his material prosperity is absolutely conditioned by his ability to apply the laws of these sciences."

This statement has the highest authority back of it for who are so well qualified to speak of the importance of a science as

its masters? Its truth will not be questioned by anyone even moderately well informed in the science if he is fair-minded.

The teacher might add that mathematics is supreme in its power to teach logical thinking to high school pupils. And in view of all these facts he might ask, "Why should not every high school pupil be required to take some form of mathematics beyond arithmetic?"

A recent editorial from the St. Louis Globe-Democrat is suggestive. The editor in commenting on the life of the late Chas. P. Steinmetz says:—

"Fortunately a distaste for mathematics did not in his day excuse from a study of that branch, as it is held by some schoolmen that it should now. Under a more robust regime his lack of inclination in that direction, extending even to an apparent inability to master the multiplication table as is said, was overcome and he became one of the foremost mathematicians of his time."

ALFRED DAVIS,
5310 Maple Ave.,
St. Louis, Mo.

## NEWS AND NOTES

The proceedings of the North Caroline Association of Colleges and Secondary Schools, Part I, 1923, contains a study of Professor C. O. Davis of the University of Michigan, on the "Size of Classes and the Teaching Load."

The final conclusions of the study are:

1. There is no necessary connection between size of class and efficiency of instruction as measured by pupils' grades.

2. Many teachers prefer to teach large classes, although the majority express themselves as preferring small or medium size classes.

3. Many pupils apparently prefer to be enrolled in large classes, although there are few data by which to be guided in making a judgment.

4. The size of a class in and of itself is not a paramount factor in determining the equity of the teaching load.

5. If teachers could be relieved of some of the added clerical duties incident to large classas, greater numbers would prefer to teach such classes, and there would, moreover, be ample justification for the administrative authorities to assign at least some large classes to all such persons.

6. The North Central Association is not justified in demanding that for *all* teachers, in *all* types of work, the maximum size of class shall be no greater than 30 pupils, or that the maximum number of pupil-hours of instruction per day shall not exceed one hundred fifty.

7. The most important determinants of the teaching load are: (a) the personality of the class; (b) the number of different preparations for class-work required daily; (c) the number of classes taught daily; (d) the amount of clerical work connected with the teaching process; (e) extra curricular and extra classroom school duties; and (f) social and civic demands.

8. "The principle of the school (to employ the exact words of one of the teachers making the reports) is the only one who can intelligently apportion the work of the school," even he, however has little scientific data to guide him.

As corrolaries of the conclusions reached above, the following practical deductions seem warranted:

1. A considerable economy of money can be had by organizing some, at least, of the classes in the school as large classes, and by putting in charge of these classes teachers who can effectively manage and instruct them.

2. Considerations of good administration demand that machinery of some sort be developed whereby teachers who are capable of instructing large classes, and prefer so to do, shall be discovered, trained, and promoted.

3. An obligation rests upon school standardizing agencies to assist in dissipating the erroneous notion that large classes are always undesirable and should be avoided.

4. The teaching load should be adjusted upon as scientific a basis as possible, but with reference always to the ability of each individual to carry burdens.

5. Promotions and financial rewards should be graded in accordance with the size and importance of the load carried.

6. An obligation rests upon school administrative officers to aid teachers in securing desirable living quarters and to assist them to find opportunities for congenial social intercourse, and suitable recreational diversions.

THE Central Association of Science and Mathematics Teachers meets in Indianapolis November 30, and December 1. The program of the Mathematics Section includes:

(1) *Geometrical Figures in Nature-Lecture and Pictures*, by Miss Edith A. Inks, Oak Park, Ill.
(2) *Study of Problems Connected with the Teaching of Mathematic*: A Committee Report, Professor W. W. Hart, Madison, Wis.
(3) *The Slide Rule in Secondary Schools*: A Committee Report, W. W. Gorsline, Chicago, Ill.
(4) *Mathematics Courses in the Senior High School*: Fiske Allen, Charleston, Ill.

The Officers of the section are: *Chairman*—E. L. Thompson, Joliet, Ill.; *Vice Chairman*—A. M. Allison, Chicago, Ill.; *Secretary*—Gertrude L. Anthony, Oak Park, Ill.

RALEIGH SCHORLING has resigned his position as teacher of mathematics in the Lincoln School of Teachers College, to become the principal of the new University of Michigan high school. Gordon R. Mirick of the Scarborough School succeeds Mr. Schorling in the Lincoln School.

W. D. Reeve, of the University of Minnesota has obtained a leave of absence for the year 1923-1924 to study in Teachers College.

The American Mathematical Society announces that it has established an honorary lectureship to be known as the Josiah Willard Gibbs Lectureship. Named for the scientist, than whom America has proposed none greater, it is proposed to invite to it at intervals of perhaps a year a distinguished investigator who shall deliver an address in semi-popular form on some topic in mathematics or its applications.

To inaugurate the series a committee has been appointed by the Society to select the first lecturer and to arrange for the lecture to be given in New York City during the winter of 1923-24.

The Association of Teachers of Mathematics in the Middle States and Maryland meets in South Bethlem, Pa., December 1, 1923. The program consists of:

1. Present Day Tendencies in Secondary School Mathematics, Dr. Jonathan Rorer, William Penn High School, Philadelphia;
2. General Mathematics,—What It Is and Who Should Take It, John R. Clark, The Lincoln School of Teachers College;
3. A Course of Study in Mathematics for Pennsylvania State Normal Schools, Miss Elsie O. Bull, Westchester State Normal, Westchester, Pa.

The officers of the Association are Dr. Florence P. Lewis, Goucher College, John C. Bechtel, Germantown High School, Philadelphia and Clarence P. Scoborio, Polytechnic Preparatory School, Brooklyn, N. Y.

Dr. William H. Metzler, dean of the College of Liberal Arts of Syracuse University, has resigned to succeed Doctor Horner as dean of the New York State College for Teachers. Dr. Metzler is a highly successful administrator and has an international reputation as a specialist in mathematics. He has been honored with membership in the leading societies devoted to mathematics not only in this country but abroad. He has been a positive force at Syracuse University through his sympathy and tact in dealing with students and his insistence upon high standards for them. *Educational Review*, Sept. 1923.

THE eighth summer meeting of the Mathematical Association of America was held at Vassar College, Poughkeepsie, N. Y., September 5-6, 1923 in conjunction with and immediately preceeding the summer meeting of the American Mathematical Society. The program included:

1. *Elliptic Geometry,*—Professor J. P. Pierpont, Yale University.
2. *The Influence of Engineering on Mathematical Teaching,*—Professor E. R. Hedrick, University of Missouri.
3. *The Honor Student in Mathematics,*—Professor Suzan R. Benedict, Smith College.
4. *A Mathematical Formulation of the Law of Growth,*—Professor L. J. Reed, John Hopkins University.
5. *Mathematical Methods in Economic Research,*—Professor C. C. Morris, Ohio State University.
6. *Mathematical Training for Laboratory Men,*—Dr. J. G. Coffin, General Laboratories, U. S. Rubber Co.
7. *Geophysics and Mathematics,*—W. D. Lambert, U. S. Coast and Geodetic Survey.
8. *Mathematics Applied and Misapplied,*—Dr. C. C. Grove, Metropolitan Life Insurance Company.
9. *An Introductory Account of the Arithmetical Theory of Algebraic Numbers and its Recent Developments,*—L. J. Mordell, Fielden Professor of Pure Mathematics, University of Manchester, England, by invitation of the American Mathematical Society.

Presidential Retiring Address: *Mathematics and Mathematicians, and Music,*—Professor R. C. Archibald, Brown University.

# NEW BOOKS

**Common Sense of the Calculus.** By G. W. Brewster. Oxford University Press, 1923, Pp. 62.

"This is not a text-book, and does not pretend to get you through any examination. Its object is to explain simply the general meaning and purpose of the methods called 'Differential and Integral Calculus.' There is a difference between the full knowledge of a subject required for its practical use and a general acquaintance with its underlying ideas. Some such acquaintance ought to be a part of everybody's education, if only because the calculus is one of the greatest triumphs of the human mind. It can be easily and quickly acquired by any one who knows a little elementary algebra. This little book is intended to give a first survey. Hence it is short and as simple as possible. Difficulties of logic are for the mathematical specialist: beginners are only confused by insistence on such details, and so get their attention drawn away from the really simple principles. These principles amount to nothing more than the special kind of addition and subtraction called integration and differentiaion, and I have tried to present these in the way that has been found intelligible and interesting to boys of fourteen. Later on, when you know what you are doing, you can easily examine the difficulties as fully as is necessary.

"I hope the book may also be of use to older readers who have felt some curiosity about the calculus, but have never had time or inclination to tackle the formidable text-books generally available. A glance at the table on the opposite page will show how little previous knowledge is really required. It is best to read the whole straight through—in fact my main object was to write a book which could be read in an easy chair. A second reading (working out some of the examples) should then give no difficulty."

**Essentials of Plane Geometry.** By David Eugene Smith. Ginn and Company, 1923. Pp. 296.

There are two types of text books that are particularly difficult to review,—the type that contains little or nothing that might be considered a contribution to the teaching of the subject, and